Aristarchus' Methods

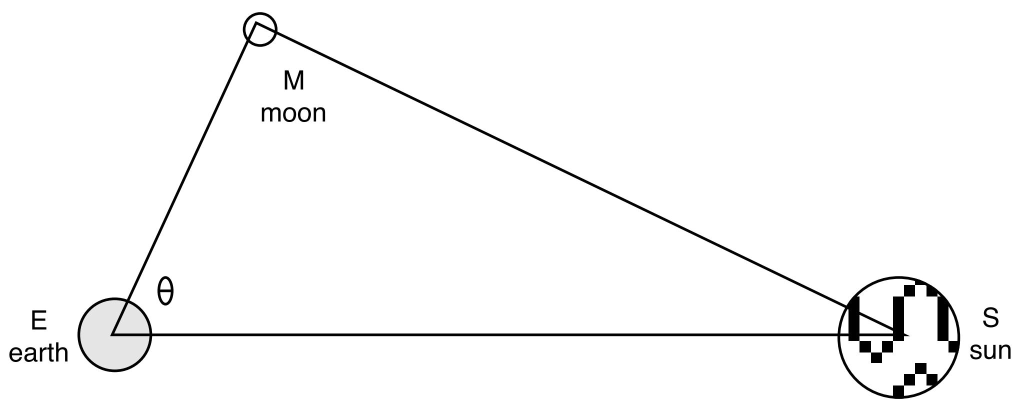
Aristarchus was able to determine the sizes of the moon and sun and how far away they are from the earth, both in terms of the size of the earth. He based his calculations on the following hypotheses/observations:

- the moon and sun are the same angular size in the sky
- when the moon is at quarter phase, the angle "earth-moon-sun" is a right angle, and he says the angle "moon-earth-sun" is 87º (in actuality, the angle is just under 90º)
- during a lunar eclipse, he measured the relative size of the moon to the shadow of the earth

We will go through the spirit of Aristarchus' achievements, but use modern math and notation. (Neither algebra, trigonometry or calculators had been invented yet; Aristarchus' work was all based on classical geometry and putting limits on the results of his calculations.) In his work, his mathematics and the ideas behind are all correct – but for some reason he uses incorrect data for the measured angles and sizes of things – so his results are in reality off. To stress that point, his methodology is correct – and as the Greeks make more accurate observations, their knowledge of the sizes and distances also becomes more accurate.

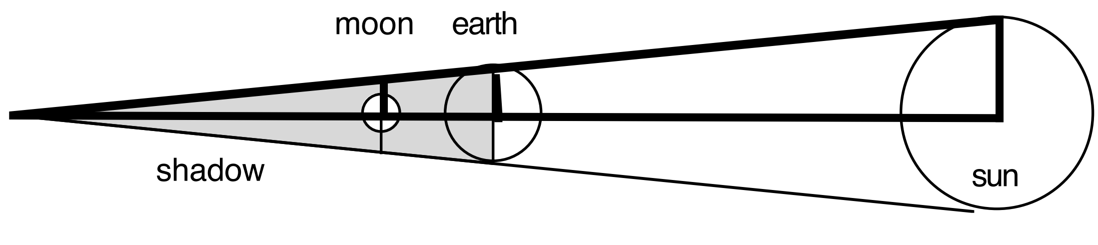
Determining the Relative Distances to the Sun and Moon

The first step Aristarchus took was to determine the relative distances to the moon and sun based on the relative positions of the sun and moon when the moon is at exactly a quarter phase (when we see the moon exactly half lit up.) When this happens, the angle EMS, shown below, is 90º. If one simply measures the angle MES, θ, then it is easy to compare the relative distances EM and ES. (In reality, this is very difficult to accomplish because it is very difficult to visually determine when the moon is exactly lit up 1/2 way.) Aristarchus used a value of 87º for ∠MES, and so found the sun to be about 20 times farther than the moon. (The angle is really just under 90º, and the sun is about 400 times farther than the moon.)



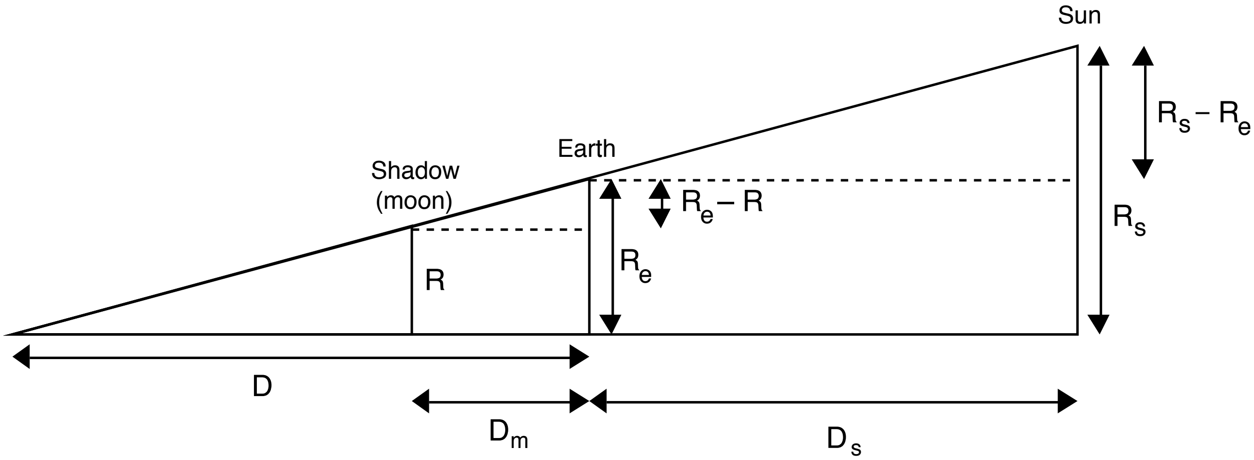
Determining the Relative Sizes of the Sun, Moon and Earth

Having calculated how much farther away the sun is compared to the moon, Aristarchus was next able to determine how much bigger the sun was compared to the earth. To do this, he needed two other pieces of information: the moon and sun are the same angular size in the sky and the size of the earth's shadow relative to the size of the moon during a lunar eclipse. The picture below shows the moon in the shadow cast by the earth during a lunar eclipse.



Aristarchus' Methods

The following diagram is taken from the previous diagram, but made a little bigger, removing the actual moon, earth and sun, and then adding a lot of labels. (The bold lines in the previous diagram are the solid lines in the one below.) As above, this is a diagram during a lunar eclipse. The big triangle is formed by the radius of the sun and the vertex of the earth's shadow. The two other altitudes drawn would represent the radius of the earth and the radius of the earth's shadow during the eclipse. (Notice that the moon itself is not part of the diagram.)



D_s = Distance to the Sun from the Earth
D_m = Distance to the Moon from the Earth
D = Length of the Earth's Shadow

R_s = Radius of the Sun
R_e = Radius of the Earth
R = Radius of the Earth's Shadow

Aristarchus' reasoning is based on a number of similar triangles from the diagram above and two other pieces of information: the size of the earth's shadow compared to the moon during a lunar eclipse, and the relative distances of the sun and moon. By looking at some similar triangles from the diagram above, and then doing a lot of algebra, we will be able to find an expression that finds the ratio of the size of the sun to the size of the earth (R_s/R_e) in terms of the relative distances to the moon and sun (found earlier) and the relative sizes of the moon and the earth's shadow during an eclipse.

Looking at the two triangles with the dotted lines as bases, they are similar, so we can say that

$$\frac{D_m}{R_e - R} = \frac{D_s}{R_s - R_e} \quad \text{rewritten as} \quad \frac{D_m}{D_s} = \frac{R_e - R}{R_s - R_e}$$

Since we know that sun and moon have the same angular size, we know that $D_m/D_s = R_m/R_s$, so we can say the following:

$$\frac{R_m}{R_s} = \frac{R_e - R}{R_s - R_e}$$

We can then rearrange the terms with a little algebra as follows:

$$\frac{R_s - R_e}{R_s} = \frac{R_e - R}{R_m}$$

$$1 - \frac{R_e}{R_s} = \frac{R_e}{R_m} - \frac{R}{R_m}$$

$$1 + \frac{R}{R_m} = \frac{R_e}{R_m} + \frac{R_e}{R_s}$$

Aristarchus' Methods

The left hand term R/R_m is measured during a solar eclipse. (Aristarchus used the value of 2 for this ratio, but that is actually a little small.) Let's take the term on the right and factor out R_e/R_s, giving us

$$1+\frac{R}{R_m}=\frac{R_e}{R_s}\left(\frac{R_s}{R_m}+1\right)$$

Which can then be rewritten to solve for R_s/R_e as

$$\frac{R_s}{R_e}=\frac{\left(1+\frac{R_s}{R_m}\right)}{\left(1+\frac{R}{R_m}\right)}$$

The two fractions on the right hand side of the equality are either measured directly, or calculated. In either case, they are known. Aristarchus said that R_s/R_m was about 19 and that R/R_m was 2, so we get

$$\frac{R_s}{R_e}=\frac{(1+19)}{(1+2)}=\frac{20}{3}$$

So Aristarchus says the sun is about 7 times the size of the earth. Since he knows the sun is about 19 times the size of the moon, he can then say that the moon is about 1/3 the size of the earth. (In reality, the moon is closer to 1/4 the size of the earth and the sun is over 100 times the size of the earth.)

Determining the Distance to the Moon

There are a couple ways you could calculate how far away the moon is: using parallax, using a lunar eclipse, or using some trig based on the size of the moon (from above) and its angular size as seen from the earth. (One of the problems in "Astronomy Problems I" uses parallax to determine how far away the moon is, so I will not discuss that here. I am not sure if Aristarchus used parallax or not, but Ptolemy did, and I would imagine that others before him did as well. Parallax would be the most accurate method of measuring how far away the moon is.)

We could also use the knowledge of how big the moon is and its angular size to determine how far away the moon is because its diameter (from above) will be approximately equal to the distance to the moon times its angular size in radians.