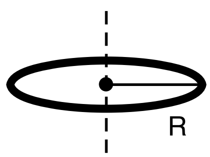
Hoop, Disc & Ring

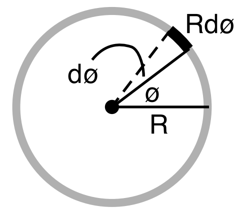
As examples, let's calculate the moments of inertia for a few objects: a hoop, a solid disc and then a ring, all of uniform density.

Hoop of mass M and radius R, axis through center



The axis of rotation is through the center of mass, and the hoop is considered to be really thin. This means that *all* of the mass of the hoop is a distance R from the axis of rotation. This means when we add up all the "mr^2" for all the little parts of the hoop, we end up just finding the total mass, and multiplying by the square of the radius. In other words, the moment of inertia is simply

$$I = MR^2$$



Let's also do this with calculus. We first imagine breaking the hoop up into lots of small pieces. The diagram to the right shows the hoop, mostly greyed out. It also shows a small piece of the hoop darker. That little piece would be our "dm", and would have a length of Rdø, where R is the radius of the hoop, and dø would be a very small change in the angle ø. The angle ø itself goes from 0 to 2π to create our hoop. Since the hoop has a uniform density, we can say that the mass per length of the hoop is M/2πR. So let's put it all together:

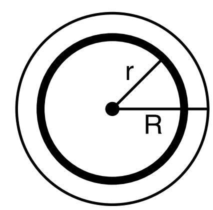
$I = \int r^2\,dm$ with $dm = (Rd\theta)\left(\frac{M}{2\pi R}\right)$ and $r^2 = R^2$, and the angle varies from 0 to 2π, so we have

$$I = \int_0^{2\pi} R^2\left(\frac{M}{2\pi R}\right)(Rd\theta) = \frac{MR^2}{2\pi}\int_0^{2\pi} d\theta$$

which becomes

$$I = \frac{MR^2}{2\pi}\theta\Big|_0^{2\pi} = MR^2$$

Disc of mass M and radius R, axis through center



To make a solid disc, we break the disc up into hoops, and simply add up the moments of inertia for all the different sized hoops. The radii of the hoops would go from 0 to R - but we need to figure out the mass of each hoop. To do that, we imagine that each hoop is very thin - dr - and then find the area of each hoop. Since the density is constant, the mass increases as the hoops get bigger.

The mass of each hoop will be the area times the density:

$$dm = (2\pi r)(dr)\left(\frac{M}{\pi R^2}\right)$$

Therefore, the moment of inertia of each hoop (dI) will be

$$dI = \left[\left(\frac{M}{\pi R^2}\right)(2\pi r)(dr)\right]r^2 = \frac{2M}{R^2}r^3dr$$

So we add up all the little moments of inertias from all the little hoops to get

$$I = \int_0^R \frac{2M}{R^2}r^3\,dr = \frac{M}{2R^2}r^4\Big|_0^R = \frac{1}{2}MR^2$$

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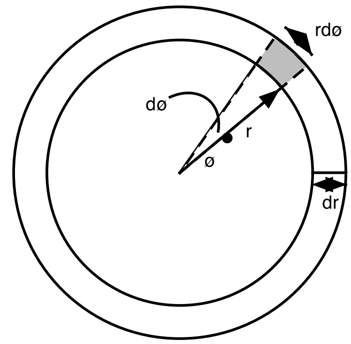
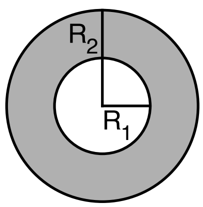


If you prefer, you can calculate the moment of inertia as a two-variable integral, and really add up all the little "mr^2" that make up the disc. The density of the disc is still obviously $M/2\pi R^2$. We need to decide the coordinate system, and in this case, it is easier to do in polar coordinates than in Cartesian. To create a disc, we vary the radius from r=0 to r=R, and then rotate it from ø=0 to ø=2π. The area of a little piece of the disc, shown shaded in the diagram to the right, would be (dr)(rdø). The mass would then be the area times the density, and the distance of each little mass to the axis of rotation would be r. So the integral looks like this:

$$I = \int r^2\, dm = \int_0^{2\pi}\int_0^{R} r^2 \left[\left(r dr d\theta \left(\frac{M}{\pi R^2}\right)\right)\right] = \frac{M}{\pi R^2}\int_0^{2\pi}\int_0^{R} r^3\, dr\, d\theta$$

Notice how it is just the two integrals we did earlier, just written down at once. It doesn't matter if you integrate over ø then r or r then ø.

Ring of mass M and inner radius R_1 and outer radius R_2, axis through center



A ring is just like making a disc - the difference being the limits of the radius. Because it is easier to write, I will do it by adding up a lot of hoops - with the moment of inertia of each hoop being mr^2. This way, we only do an integral with one variable r.

The density of the ring is easy, but messy: $M/(\pi R_2^2 - \pi R_1^2)$, where R_1 is the inner radius and R_2 is the outer radius. The setup is basically the same as that of the disc, just with a different density and different limits of integration:

$$I = \int_{R_1}^{R_2}\left[\left(\frac{M}{\pi R_2^{\,2} - \pi R_1^{\,2}}\right)(2\pi r)dr\right] r^2 = \frac{2M}{R_2^{\,2} - R_1^{\,2}}\int_{R_1}^{R_2} r^3\, dr$$

The term in the square brackets is the mass of the small hoop of radius r. This turns into

$$I = \left(\frac{2M}{R_2^{\,2} - R_1^{\,2}}\right)\frac{r^4}{4}\Bigg|_{R_1}^{R_2} = \left(\frac{M}{2(R_2^{\,2} - R_1^{\,2})}\right)\left(R_2^{\,4} - R_1^{\,4}\right)$$

So that

$$I = \frac{1}{2}M\left(R_2^{\,2} + R_1^{\,2}\right)$$