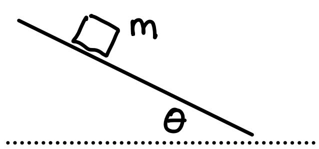
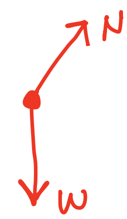
Inclined Planes & Coordinates

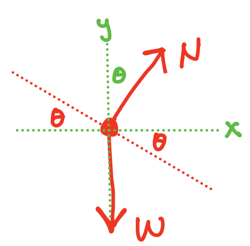


What is the acceleration of a box of mass m on a frictionless ramp with base angle θ?



The first step in using Newton's Second Law is <u>always</u> drawing a Free Body Diagram, shown to the right. There are only two forces on the mass - its weight and the normal force from the ramp.

N2L is easy - its just $\Sigma \vec{F} = m\vec{a}$ but in dealing with the fact that we have vectors, we need to choose a coordinate system. We'll do a regular x & y coordinate first (and later do a rotated system that is actually much easier.)



The free body diagram is redrawn to the left - this time with a dashed red line to represent the ramp an dashed green lines to show the x & y axis.

The angle θ (in red) is just the given base angle.

By definition, the normal force is $\perp$ to the ramp and x is $\perp$ to y. So that means the angle between the normal force and the y-axis is also θ (shown in green.)

That means the two forces would be given by:

$$\vec{N} = N\sin\theta\,\hat{\imath} + N\cos\theta\,\hat{\jmath} \quad \& \quad \vec{W} = 0\hat{\imath} - mg\hat{\jmath}$$

Since the box accelerates down the ramp, its acceleration in component form would be $\vec{a} = a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j}$

So N2L becomes

$$\Sigma\vec{F} = m\vec{a}$$

$$(N\sin\theta\,\hat{i} + N\cos\theta\,\hat{j}) + (0\,\hat{i} - mg\,\hat{j}) = m(a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j})$$

Breaking that up into the separate components we have

$\hat{i})\quad N\sin\theta = ma\cos\theta$

$\hat{j})\quad N\cos\theta - mg = -ma\sin\theta$

This is just two equations (one for each component) and two unknowns (N and a). So let's solve!

From the horizontal component we have $N = \dfrac{ma\cos\theta}{\sin\theta}$

Plugging into the vertical equation and then solving gives us.

$$\left(\frac{ma\cos\theta}{\sin\theta}\right)\cos\theta - mg = -ma\sin\theta$$

$$\cancel{m}a\cos^2\theta - \cancel{m}g\sin\theta = -\cancel{m}a\sin^2\theta$$

$$a\cos^2\theta + a\sin^2\theta = g\sin\theta$$

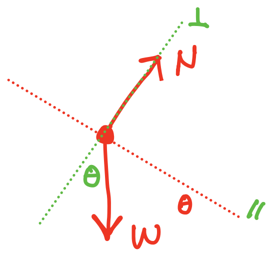
$$a(\cos^2\theta + \sin^2\theta) = g\sin\theta$$

$$a = g\sin\theta$$

Notice the units work out ($\sin\theta$ is dimensionless) and if the "ramp" was actually flat, $\theta = 0$ and so $a = g\sin(0) = 0$ which is expected. If the ramp were vertical, then $\theta = 90$ and $a = g\sin(90) = g$, which is also expected.

While that wasn't too bad, once we throw in some more forces from friction or other masses then it can get pretty involved. There is a way that will result in a lot less algebra, but you have think creatively. That means use a rotated coordinate system.

Because we know the mass accelerates down the hill, let's use the acceleration to define our coordinates: Parallel to the ramp and perpendicular to the ramp.



The free body diagram is again shown to the left, but this time the green dashed line shows ⊥ to the ramp. Notice the base angle θ is then the angle between the weight and the ⊥ axis (shown in green.)

Notice this time the Normal force is only in the ⊥ direction, while the weight has components in both the ⊥ & // directions. More importantly, the acceleration is only in the // direction.

Finally we can write out N2L

$$\Sigma \vec{F} = m\vec{a} \rightarrow \Sigma F_{//} = ma \quad \& \quad \Sigma F_{\perp} = 0$$

From the free body diagram we can finally write out

$$\Sigma F_{//} = ma \qquad \& \qquad \Sigma F_{\perp} = 0$$

$$mg\sin\theta = ma \qquad\qquad N - mg\cos\theta = 0$$

$$a = g\sin\theta$$

That is so much easier it should be criminal!

But notice we didn't need this to find the acceleration!

TLDR:

If an object is accelerating on an inclined plane, find the components of the vectors that are $\parallel$ & $\perp$ to the inclined plane. Then you will say:

$$\Sigma F_{\parallel} = ma \quad \& \quad \Sigma F_{\perp} = 0$$

Note that for a base angle θ, the components of gravity are:

$$w_{\parallel} = mg\sin\theta \quad \& \quad w_{\perp} = mg\cos\theta$$