Gravitational Field of a Sphere

Newton's gravitational theory states that you are attracted to every particle that makes up the earth. He was able to show that the gravitational attraction to a uniform sphere (almost the earth) was as if the whole mass of the sphere was at the center of mass of the sphere. We will do a version of that proof here.

To do this proof, we will first derive an expression for the gravitational attraction between an object and a uniform ring, then to a uniform spherical shell, and finally to a uniform sphere.

Uniform Ring

First we will derive an expression for the gravitational attraction between an object of mass m and a uniform ring of mass M and radius R. The object is located a distance r from the center of the ring on the central axis of the ring, as shown in figure 1.

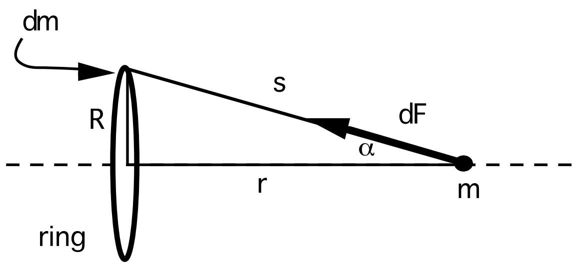

figure 1

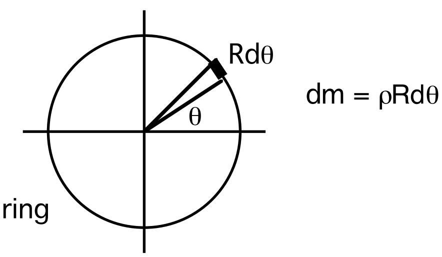

figure 2

R = radius of ring
M = mass of ring
m = mass of second object
r = distance between center of ring and second object
dm = small chunk of the ring
s = distance from mass, m, to small chunk of ring, dm
α = angle from mass , m, to small chunk of ring, dm,

First, imagine dividing up the ring into a lot of little tiny chunks (dm.) To calculate the force of attraction between the whole ring and the mass, we need to take the vector sum of the forces between the mass, m, and each little chunk of ring, dm. To do this, we look at the symmetry of the arrangement, and notice that for every point on the ring, there is a point exactly opposite it. The non-axial components of each chunk of force cancel out. All we need to add up are the axial components of the attraction.

So, the axial component of the force between the mass and the chunk of the ring is:

$$dF = G\frac{mdm}{s^2}\cos\alpha$$

Now we need to set up the integration. We will integrate around the ring, from θ = 0 to 2π. (See figure 2.) We need an expression for the chunk of mass as it depends on θ. Defining the linear density of the ring as

$$\rho = \frac{M_{ring}}{C_{ring}} = \frac{M}{2\pi R}$$

we can say that

$$dm = \rho R d\theta$$

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The integral thus follows:

$$
\begin{aligned}
F &= \int dF \\
&= \int G \frac{mdm}{s^2} \cos\alpha \\
&= \int_0^{2\pi} G \frac{m(\rho R d\theta)}{s^2} \cos\alpha \\
&= G \frac{m\rho R}{s^2} \cos\alpha \int_0^{2\pi} d\theta \\
&= G \frac{m\rho 2\pi R}{s^2} \cos\alpha \\
&= G \frac{mM}{s^2} \cos\alpha
\end{aligned}
$$

We will use the above result in the integration for the spherical shell, but let's continue with some substitutions to make it use less variables. From the original diagram we have the following identities

$$s = \sqrt{r^2 + R^2}$$

$$\cos\alpha = \frac{r}{s} = \frac{r}{\sqrt{r^2 + R^2}}$$

Substituting these into our result leaves the following expression:

$$\boxed{F = G \frac{mM}{\left(r^2 + R^2\right)^{3/2}} r}$$

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Uniform Spherical Shell

To calculate the gravitational attraction between an object of mass m and a spherical shell of mass M and radius R a distance of r from the mass, we will divide the shell into a bunch of rings, and simply add up the force between each ring and the mass. Figure 3 shows a ring on the sphere and figure 4 shows a side view with clearer labels. (The letters represent the same things from the previous derivation.)

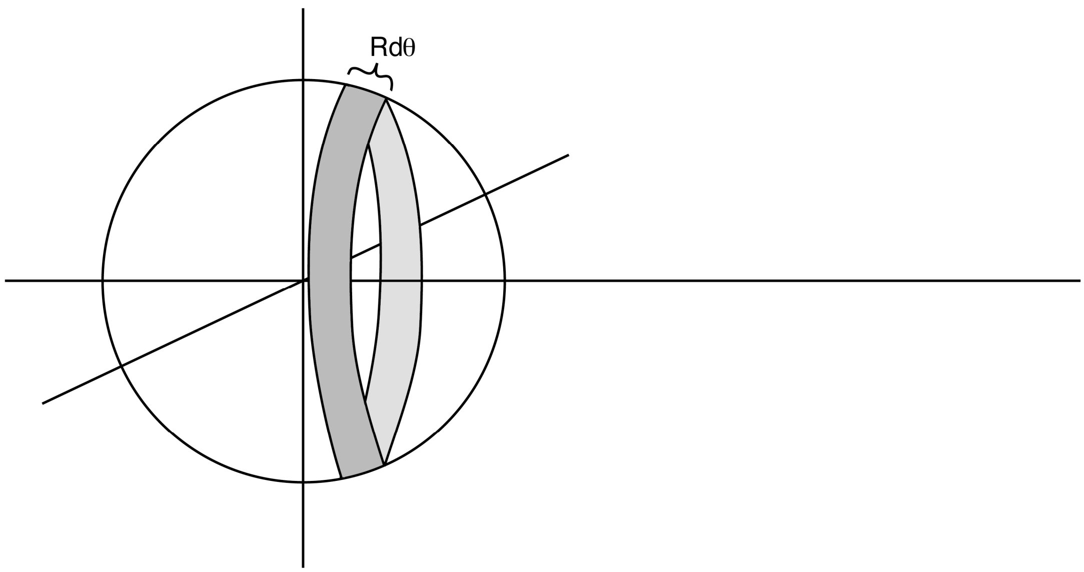

Figure 3

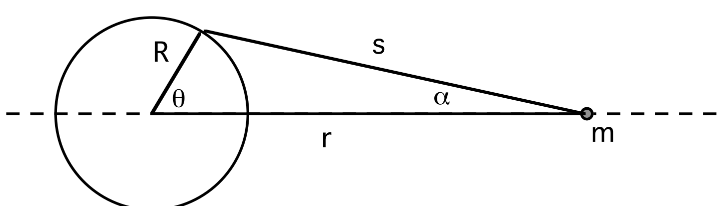

figure 4

From the previous section, we know that the attraction to a ring will be

$$dF = G\frac{mM_{ring}}{s^2}\cos\alpha$$

We need to find an expression for the mass of each ring as it depends on θ. We define the surface density of the sphere as

$$\sigma = \frac{M_{sphere}}{A_{sphere}} = \frac{M}{4\pi R^2}$$

So the mass of a ring would be

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$$density \times circumference \times width$$
$$dm = \sigma(2\pi R\sin\theta)(Rd\theta)$$
$$dm = 2\pi R^2\sigma\sin\theta d\theta$$

Now we have the rather unsightly differential to integrate from θ = 0 to 2π

$$dF = G\frac{m2\pi R^2\sigma\sin\theta d\theta}{s^2}\cos\alpha$$

Unfortunately, we cannot do this integral yet. The variables s and α vary with θ. While it would appear natural to simply find expressions for s and α that depend on θ (since that is our differential), that will give an integral that I do not know how to do. (At which point I would simply look it up in the CRC handbook.) We will derive expressions for θ and α that depend on s.

From the law of cosines we can write an expression for cosα:

$$R^2 = s^2 + r^2 - 2sr\cos\alpha$$
$$\cos\alpha = \frac{s^2 + r^2 - R^2}{2sr}$$

To get rid of the 2 terms involving θ, we will again use the law of cosines:

$$s^2 = r^2 + R^2 - 2rR\cos\theta$$

Differentiating this gives us:

$$2sds = 2rR\sin\theta d\theta$$

Which we rewrite as:

$$R\sin\theta d\theta = \frac{sds}{r}$$

Finally, we are able to set up the integral, make the substitutions, and solve.

$$\begin{aligned} dF &= G\frac{m2\pi R\sigma}{s^2}(R\sin\theta d\theta)(\cos\alpha) \\ &= G\frac{m2\pi R\sigma}{s^2}\left(\frac{sds}{r}\right)\left(\frac{s^2 + r^2 - R^2}{2sr}\right) \\ &= G\frac{m\pi R\sigma}{r^2}\left(\frac{s^2 + r^2 - R^2}{s^2}\right)ds \end{aligned}$$

In changing variables, the limits of the integral also change. Instead of integrating from θ = 0 to 2π, we are integrating from s = r-R to r+R. So the integral is

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$$F = \int_{r-R}^{r+R} G\frac{m\pi R\sigma}{r^2}\left(\frac{s^2 + r^2 - R^2}{s^2}\right)ds$$
$$= G\frac{m\pi R\sigma}{r^2}\int_{r-R}^{r+R}\left(1 + \frac{r^2 - R^2}{s^2}\right)ds$$

Which becomes:

$$F = G\frac{m\pi R\sigma}{r^2}\left[s - \frac{r^2 - R^2}{s}\right]_{r-R}^{r+R}$$
$$= G\frac{m\pi R\sigma}{r^2}\left[\left((r+R) - \frac{r^2 - R^2}{(r+R)}\right) - \left((r-R) - \frac{r^2 - R^2}{(r-R)}\right)\right]$$
$$= G\frac{m\pi R\sigma}{r^2}[4R]$$
$$= G\frac{m4\pi R^2\sigma}{r^2}$$

Substituting in our definition of σ, we finally arrive at:

$$\boxed{F = G\frac{mM}{r^2}}$$

The above result is for an object outside the sphere. To calculate the attraction to a point *inside* the sphere, we have figure 5.

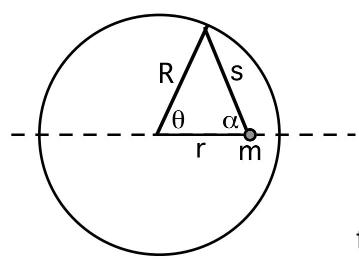

figure 5

The integral is set up the exact same way, with only one difference: the limits of the integration are from s = R-r to R+r. (To see this, look at what happens to s in figure 5 when we make θ change from 0 to 2π.)

The integral thus becomes:

$$F = G\frac{m\pi R\sigma}{r^2}\left[s - \frac{r^2 - R^2}{s}\right]_{R-r}^{R+r}$$
$$= G\frac{m\pi R\sigma}{r^2}\left[\left((R+r) - \frac{r^2 - R^2}{(R+r)}\right) - \left((R-r) - \frac{r^2 - R^2}{(R-r)}\right)\right]$$
$$= G\frac{m\pi R\sigma}{r^2}[0]$$
$$= 0$$

The gravitational force on an object inside a uniform spherical shell is zero!

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Uniform Sphere

To calculate the gravitational attraction to a uniform sphere, we simply treat the sphere as bunch of shells, and add up the force from all the shells. In figure 6, x is the radius of a shell. All other letters represent the same things they did earlier.

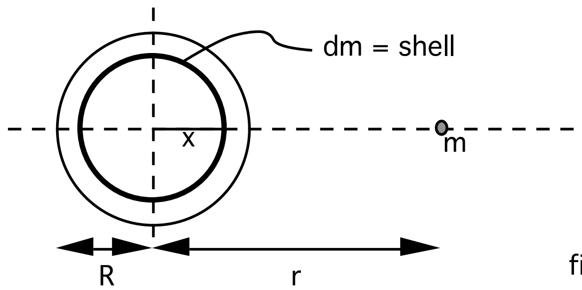

figure 6

Defining the density of the sphere to be

$$\rho = \frac{M_{sphere}}{V_{sphere}} = \frac{M}{\frac{4}{3}\pi R^3}$$

We have the mass dm of a shell:

$$density \times area \times thickness$$

$$dm = \rho\left(4\pi x^2\right)dx$$

Setting up the integral, and integrating from x = 0 to R gives us

$$\begin{aligned} F &= \int dF \\ &= \int G\frac{mdm}{r^2} \\ &= \int_0^R G\frac{m\left(4\pi\rho x^2\right)}{r^2}dx \\ &= G\frac{m4\pi\rho}{r^2}\int_0^R x^2 dx \\ &= G\frac{m4\pi\rho}{r^2}\left[\tfrac{1}{3}x^3\right]_0^R \\ &= G\frac{m\frac{4}{3}\pi R^3\rho}{r^2} \end{aligned}$$

Finally, substituting in our expression for the density of the sphere, we get:

$$\boxed{F = G\frac{mM}{r^2}}$$

As was the case for the shells, the gravitational attraction to a uniform sphere is as if the whole sphere were one particle at the center of the sphere and of the same mass as the sphere.